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Intersection numbers on Grassmannians, and on the space of holomorphic maps from CP^1 into $G_r(C^n)$

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Abstract

We derive some explicit expressions for correlators on Grassmannian $G_r(C^n)$ as well as on the moduli space of holomorphic maps, of a fixed degree d, from sphere into the Grassmannian. Correlators obtained on the Grassmannian are a first-step generalization of the Schubert formula for the self-intersection. The intersection numbers on the moduli space for $r = 2, 3$ are given explicitly by two closed formulas, when $r = 2$ the intersection numbers are found to generate the alternate Fibonacci numbers, the Pell numbers and in general a random walk of a particle on a line with absorbing barriers. For $r = 3$, the intersection numbers form a well-organized pattern.

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1. Introduction

The classical Schubert calculus computes intersection numbers on Grassmannians $G_r(C^n)$ of complex r-planes in C^n by using the Giambelli and Pieri formula [1–3]. It is due primarily to Schubert more than a 100 years ago to obtain the number,

$$\int_{G_r(C^n)} x_1^{r(n-r)} = \frac{1!2!3!\cdots(r-2)!(r-1)!(r(n-r))!}{(n-r)!(n-r+1)!\cdots(n-1)!},$$

known as the degree of the Grassmannian or the self-intersection of the first Chern class x_1, of the r-plane bundle Q on $G_r(C^n)$. Geometrically speaking this number corresponds

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to the number of $(r-1)$-planes in CP^{n-1} meeting $r(n-r)$ general $(n-r-1)$-planes, in particular for $r=2$ there are two lines meeting 4-given lines in CP^3.

Our goals in this paper are twofold, first we would like to extend the above formula to other correlators that are products of Chern classes x_i, $1 \le i \le r$. In this direction, we use the pairing residue formula that computes the correlators in topological Landau–Ginzburg theories [4] and the explicit formula for the potential $W(x_i)$ that generate the cohomology ring [5] to do some explicit computations on the Grassmannian. The different intersection numbers obtained show a certain pattern amongst themselves and is formulated in a proposition which in turn lead to the closed formula for $\int_{G_r(C^n)} x_1^{r(n-r)-rk_{r-1}} x_r^{k_{r-1}}$.

The second goal is to carry out similar computations on the space of holomorphic maps of a fixed degree d from a Riemann surface of genus zero (CP^1) into the Grassmannian $G_r(C^n)$. Formally, both computations use the same formula [4,6], the difference between the two cases is that the potential in the second case is a deformed potential $\tilde{W}(x_i)$ and is connected to the previous potential by $\tilde{W}(x_i) = W(x) + (-1)^r q x_1$ [7]. This potential reproduces the quantum cohomology ring of the Grassmannian [8]. The concept of deformation of the cohomology ring was first observed in [9] in connection with the CP^1 model. On the space of holomorphic maps from CP^1 into $G_r(C^n)$, we have two closed formulas for $r=2$ and 3 for any n. When $r=2$, the intersection numbers generate well-known numbers like the Fibonacci numbers and the Pell numbers for $n=5,6$, respectively, and when $n \ge 7$, the intersection numbers generate a random walk of a particle on a line with absorbing barriers [10,11]. Our closed formula for the intersection numbers on the space of holomorphic maps into $G_3(C^n)$, when restricted to constant maps, gives all the intersection numbers on $G_3(C^n)$. Some intersection numbers on this space were computed when $n=6$ and for degrees 1 and 2. We find that these numbers organize themselves in an ordered pattern, it seems that the intersection numbers on this moduli space is a zoo of interesting numbers. This fact is already presented on the Grassmannian $G_r(C^n)$; if we set $N = n-r$ in the Schubert formula, we obtain the generating functions for the r-dimensional Catalan numbers [10]. When $r=2$, we obtain the ordinary Catalan numbers $(2N)!/(N+1)!N!$

This paper is organized as follows. In Section 2, after a brief account of the cohomology ring of the Grassmannian, the pairing residue formula which computes correlators in $N=2$ topological Landau theories, and fixing our notations, we compute some intersection number on the Grassmannian $G_r(C^n)$ from which we obtain closed formula for the correlators. Sections 3 and 4 will be devoted to computations of correlators on the space of holomorphic maps from CP^1 into $G_r(C^n)$, in which we find connections between intersection numbers, Fibonacci numbers, Pell numbers and the random walk. Our conclusions are given in Section 5.

2. Intersection numbers on a Grassmannian

In this section, we shall first recall briefly the definition of the cohomology ring of the Grassmannian $G_r(C^n)$ in the Landau–Ginzburg formulation [9,12,13] and the pairing residue formula of the $N=2$ topological Landau–Ginzburg model that computes the correlators [4]. We then use the specialized form of "the pairing residue formula" and the explicit expression for the Landau–Ginzburg potential in terms of the generators $x_i (1 \leq i \leq r)$ of the cohomology ring of the Grassmannian [5] to compute some intersection numbers. The intersection numbers computed are exactly those obtained using the Schubert calculus [2,3]. These computations show that the two-point function $\langle x_1^{\alpha_1} x_2^{\alpha_2} \rangle$ on $G_2(C^{n+1})$ is equal to the two-point function $\langle x_1^{\alpha_1} x_2^{\alpha_2 - 1} \rangle$ on $G_2(C^n)$. In general, the r-point functions on $G_r(C^{n+1})$ and $G_r(C^n)$ are related in the same way. This fact will be proved in the proposition below and as a consequence we obtain an explicit expression for the two-point functions on $G_r(C^n)$ involving the Chern classes x_1 and x_r.

The cohomology ring of the complex Grassmannian manifold, denoted by $H^*(G_r(C^n))$ is a truncated polynomial ring in several variables [14] given by

$$H^*(G_r(C^n)) \cong \frac{C[x_1, \dots, x_r, y_1, \dots, y_{n-r}]}{I}, \tag{1}$$

where $x_i = c_i(Q)$ (for $1 \leq i \leq r$) are the Chern classes of the quotient bundle Q of rank r, i.e., $x_i \in H^{2i}(G_r(C^n))$ and $y_j = c_j(S)$ (for $1 \leq j \leq n - r$) are the Chern classes of the universal bundle S of rank $n - r$. The ideal I in $C[x_1, \dots, x_r, y_1, \dots, y_{n-r}]$ is given by

$$(1 + x_1 + x_2 + \cdots + x_r)(1 + y_1 + y_2 + \cdots + y_{n-r}) = 1, \tag{2}$$

which is the consequence of the tautological sequence on $G_r(C^n)$

$$0 \to S \to V \to Q \to 0,$$

where $V = G_r(C^n) \times C^n$. By using Eq. (2), one may rewrite $H^*(G_r(C^n))$ as

$$H^*(G(C^n)) \cong \frac{C[x_1, \dots, x_r]}{y_j}, \tag{3}$$

where y_j are expressed in terms of x_i, and $y_j = 0$ for $n - r + 1 \leq j \leq n$, and $x_0 = y_0 = 1$. The classes y_j can be written inductively as a function of $x_1, \dots, x_r$ via

$$y_j = -x_1 y_{j-1} - \cdots - x_{j-1} y_1 - x_j \quad \text{for } j = 1, \dots, n - r. \tag{4}$$

In the Landau–Ginzburg formulation, the potential that generates the cohomology ring of the Grassmannian [9,12,13] is given by

$$W_{n+1}(x_1, \dots, x_r) = \sum_{i=1}^{r} \frac{q_i^{n+1}}{n+1}, \tag{5}$$

where x_i and q_i are related by

$$x_i = \sum_{1 \leq l_1 < l_2 \cdots < l_i \leq r} q_{l_1} q_{l_2} \cdots q_{l_i}. \tag{6}$$

The cohomology ring of the Grassmannian is then given by

$$\frac{\partial W_{n+1}}{\partial x_i} = (-1)^n y_{n+1-i} \quad \text{for } 1 \leq i \leq r, \tag{7}$$

implying that $\mathrm{d}_i W_{n+1} = 0$ for $i = 1, \dots, r$. In terms of the x_i's [5], the explicit formulas for the y_j's and the cohomology potential $W(x_1, \dots, x_r)$ are

$$y_j = (-1)^j \sum_{k_1=0}^{[j/2]} \cdots \sum_{k_{r-1}=0}^{[j/r]} \frac{(-1)^{k_1+2k_2+\cdots+(r-1)k_{r-1}}}{k_1! \cdots k_{r-1}!} \times \frac{\left(j - \sum_{l=1}^{r-1} l k_l\right)!}{\left(j - \sum_{l=2}^{r} l k_{l-1}\right)!} x_1^{j-2k_1-\cdots-rk_{r-1}} x_2^{k_1} \cdots x_r^{k_{r-1}}, \tag{8}$$

$$W_{n+1}(x_1, \dots, x_r) = \sum_{k_1=0}^{[n+1/2]} \cdots \sum_{k_{r-1}=0}^{[n+1/r]} \frac{(-1)^{k_1+2k_2+\cdots+(r-1)k_{r-1}}}{k_1! \cdots k_{r-1}!} \times \frac{\left(n - \sum_{j=1}^{r-1} j k_j\right)!}{\left(n + 1 - \sum_{j=2}^{r} j k_{j-1}\right)!} x_1^{n+1-2k_1-\cdots-rk_{r-1}} x_2^{k_1} \cdots x_r^{k_{r-1}}. \tag{9}$$

The self-intersection numbers $\langle x_1^{r(n-r)} \rangle$, and other correlation functions on the Grassmannian $G_r(C^n)$ of products of monomials in the cohomology classes $x_i (1 \le i \le r)$ such that the total power of this product is the dimension of $G_r(C)$, i.e., $r(n-r)$, may be computed using the residue pairing formula [4]. This formula computes the correlators in the topological Landau–Ginzburg theories, which for genus zero, reads

$$\left\langle \prod_{i=1}^{N} F_i(x_j) \right\rangle = (-1)^{N(N-1)/2} \sum_{\mathrm{d}W=0} \frac{\prod_{i=1}^{N} F_i(x_j)}{H} = (-1)^{N(N-1)/2} \frac{1}{(2\pi i)^N} \oint \cdots \oint \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_N \prod_{i=1}^{N} F_i(x_j)}{\partial_1 W \cdots \partial_N W}, \tag{10}$$

where $F_i(x_j)$ are polynomials in the superfields x_i, $H = \det(\partial_i \partial_j W)$ is the Hessian and the summation on the right-hand side in the first expansion is over the critical points of W. In this section, the maps from sphere into $G_r(C^n)$ are considered constant, i.e., the moduli space of instantons is nothing but the Grassmannian $G_r(C^n)$ itself, and the correlators are the intersections of the cycles over $G_r(C^n)$. Therefore, the residue pairing formula reads [1]

$$\langle x_1^{n+1-2k_1-\cdots-rk_{r-1}} x_2^{k_1} \cdots x_r^{k_{r-1}} \rangle = \frac{(-1)^{r(r-1)/2}}{(2\pi i)^r} \oint \cdots \oint \frac{x_1^{n+1-2k_1-\cdots-rk_{r-1}} x_2^{k_1} \cdots x_r^{k_{r-1}}}{\partial_1 W \cdots \partial_r W}. \tag{11}$$

The closed form for these correlators is, in general, not known, except for the self-intersection $\langle x_1^{r(n-r)} \rangle$, which was given by the Schubert calculus [1–3] (also called the degree of the

[1] This is a natural parameterization for the powers of x_i's since the total power sums up to $r(n-r)$, otherwise the correlators vanish.

Table 1
Intersection numbers I_k^n on $G_2(C^n)$ for $n = 4, 5, 6, 7$

n	k					
	0	1	2	3	4	5
4	2	1	1			
5	5	2	1	1		
6	14	5	2	1	1	
7	42	14	5	2	1	1

Grassmannian) and has the following expression:

$$\langle x_1^{r(n-r)}\rangle = (r(n-r))!\prod_{\ell=0}^{r-1}\frac{\ell!}{(n-r+\ell)!}. \tag{12}$$

In particular for $G_2(C^4)$, the self-intersection $\langle x_1^4\rangle$, is 2 which is the number of lines meeting 4 given lines in CP^1, and in general the right-hand side of the above equation gives the number of $(r-1)$-planes meeting $N = r(n-r)$ given $(n-(r+1))$-planes in general position in CP^{n-1}. The simplest non-trivial Grassmannian for which the residue pairing formula can be used is $G_2(C^4)$. Here the potential that generates the cohomology ring (intersection ring) $H^*(G_2(C^4))$ is $W(x_1, x_2) = \frac{1}{5}x_1^5 - x_1^3x_2 + x_1x_2^2$ and the possible correlators are $\langle x_1^{4-2k}x_2^k\rangle$, where $0 \le k \le 2$. Applying the residue pairing formula, we have

$$\langle x_1^{4-2k}x_2^k\rangle = -\frac{1}{(2\pi i)^2}\oint\oint\frac{x_1^{4-2k}x_2^k\,\mathrm{d}x_1\,\mathrm{d}x_2}{(x_1^4-3x_1^2x_2+x_2^2)(-x_1^3+2x_1x_2)}. \tag{13}$$

Explicit computation for $k = 0, 1, 2$ gives ($\partial_2 w = 0$ for $x_2 = \frac{1}{2}x_1^2$)$\langle x_1^4\rangle = 2$, $\langle x_1^2x_2\rangle = 1$ and $\langle x_2^2\rangle = 1$, which agree with the Schubert calculus [3]. In the same way, we have computed the correlators $I_k^n := \langle x_1^{2(n-2)-2k}x_2^k\rangle$ for $n = 5, 6, 7$, on $G_2(C^n)$ and $I_{k_1,k_2}^n := \langle x_1^{3(n-3)-2k_1-3k_2}x_2^{k_1}x_3^{k_2}\rangle$ on $G_3(C^n)$ for $n = 5, 6, 7$ and the results obtained are indicated in Tables 1 and 2.

We have checked our computations using the property, $Res_W(H) = \mu$, where μ is the criticality index of W [4], i.e., the dimension of chiral ring $R = C[x_i]/\mathrm{d}W_i$. The above computations on $G_2(C^n)$ and $G_3(C^n)$ indicate that we should have $\langle x_1^{\alpha_1}x_2^{\alpha_2}\rangle_{G_2(C^{n+1})} = \langle x_1^{\alpha_1}x_2^{\alpha_2-1}\rangle_{G_2(C^n)}$, $\langle x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\rangle_{G_3(C^{n+1})} = \langle x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3-1}\rangle_{G_3(C^n)}$ and, in general,

$$\langle x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_r^{\alpha_r}\rangle_{G_r(C^{n+1})} = \langle x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_r^{\alpha_r-1}\rangle_{G_r(C^n)}$$

with $\sum_{i=1}^{r-1}\alpha_i = r(n-r)$. This is indeed the case as we shall show in the proposition below; but first we need the following lemma.

Lemma 1. *Given an inclusion $i : Z \hookrightarrow X$ (non-singular subvariety) with $\dim_C X - \dim_C Z = r = n - m$ and suppose there exists a complex vector bundle E on X such that $E|_Z = N_{Z,X}$ (normal bundle of Z in X) and $\alpha \in H^{2m}(X) = H^{2n-2r}(X)$, then $i^*(\alpha) = \alpha X_r(E)$.*

Table 2
Intersection numbers $I^n_{k_1,k_2}$ on $G_3(C^n)$ for $n = 5, 6, 7$

(k_1, k_2)	$I^5_{k_1,k_2}$	(k_1, k_2)	$I^6_{k_1,k_2}$	(k_1, k_2)	$I^7_{k_1,k_2}$
(0, 0)	5	(0, 0)	42	(0, 0)	462
(1, 0)	3	(1, 0)	21	(1, 0)	210
(2, 0)	2	(2, 0)	11	(2, 0)	98
(3, 0)	1	(3, 0)	6	(3, 0)	47
(1, 1)	1	(0, 1)	5	(0, 1)	42
(0, 1)	1	(4, 0)	3	(4, 0)	23
(0, 2)	1	(1, 1)	3	(1, 1)	21
		(2, 1)	2	(5, 0)	11
		(0, 2)	1	(2, 1)	11
		(1, 2)	1	(3, 1)	6
		(0, 3)	1	(6, 0)	5
				(0, 2)	5
				(4, 1)	3
				(1, 2)	3
				(2, 2)	2
				(0, 3)	1
				(0, 4)	1
				(1, 3)	1
				(3, 2)	1

For simplicity, consider the case $G_2(C^n) \hookrightarrow G_2(C^{n+1})$, and let ${}^{n+1}Q$ and ${}^nQ$ denote the quotient subbundles on $G_2(C^{n+1})$ and $G_2(C^n)$, respectively, both of rank 2. Then, the induced pullback gives $i^*({}^{n+1}Q) = {}^nQ$, furthermore ${}^{n+1}Q|_{G_2(C^n)} = N_{G_2(C^n),G_2(C^{n+1})}$. The above remarks on the intersection numbers on $G_2(C^n)$, $G_2(C^{n+1})$ computed by the residue pairing formula are equivalent to the following proposition.

Proposition 1. *The correlators on $G_2(C^{n+1})$ and $G_2(C^n)$ are identical in the following sense*:

$$\langle x_1({}^{n+1}Q)^{2n-4-2k} x_2({}^{n+1}Q)^{k+1}\rangle = \langle x_1({}^nQ)^{2n-4-2k} x_2({}^nQ)^k\rangle. \tag{14}$$

Proof. Setting $x_1({}^nQ)^{2n-4-2k} = U$, $x_1({}^{n+1}Q)^{2n-4-2k} = U'$, $x_2({}^nQ)^k = V^k$ and $x_2({}^{n+1}Q)^k = V'^k$, applying the above lemma; $i^*(\alpha) = \alpha x_r({}^{n+1}Q)$ with $r = 2$ then $i^*(U'V'^k) = (U'V'^k)V' = U'V'^{k+1}$ and by using the homomorphism of the pullback, the left-hand side is *UV*, hence the proof of the proposition. □

The above proposition can be generalized to correlators on $G_r(C^n)$, namely, we will have the following:

$$\begin{aligned}&\langle x_1({}^{n+1}Q)^{r(n-r)-2k_1-3k_2-\cdots-rk_{r-1}} x_2({}^{n+1}Q)^{k_1}\cdots x_r({}^{n+1}Q)^{k_{r-1}}\rangle\\ &\quad= \langle x_1({}^nQ)^{r(n-r)-2k_1-3k_2-\cdots-rk_{r-1}} x_2({}^nQ)^{k_1}\cdots x_r({}^nQ)^{k_{r-1}-1}\rangle.\end{aligned} \tag{15}$$

As a consequence of the proposition, we have a closed formula for the two-point functions on $G_r(C^n)$, containing x_1 and x_r given by

$$\langle x_1^{r(n-r)-rk_{r-1}} x_r^{k_{r-1}} \rangle = (r(n-k_{r-1}-r))! \prod_{\ell=0}^{r-1} \frac{\ell!}{(n-k_{r-1}-1+\ell)!}, \tag{16}$$

which is obtained from the self-intersection formula equation (12) simply by the shift $n \to n - k_{r-1}$. In particular, on $G_2(C^n)$, we have the following closed formula:

$$\langle x_1^{2n-4-2k} x_2^k \rangle = \frac{(2(n-2-k))!}{(n-k-2)!(n-k-1)!}. \tag{17}$$

This particular case, that we denote by I_k^n, was also obtained using topological Kazama–Suzuki models based on complex Grassmannian [15]. If we set $k_{r-1} = n - r$ in Eq. (16), one obtains $\langle x_r^{n-r} \rangle = \int_{G_r(C^n)} x_r^{n-1} = 1$ as was shown in [9]. The closed formula given by Eq. (16) is consistent with the proposition above, since the formula is automatically invariant under the shifts $n \to n+1$ and $k_r \to k_r + 1$, which in turn gives the two-point function on $G_r(C^{n+1})$.

3. Intersection numbers on the space of holomorphic maps to a Grassmannian

Here and in the next section we will give two explicit formulas for the intersection numbers on the space of holomorphic maps of degree d from CP^1 into $G_r(C^n)$ for $r = 2, 3$. This space of maps is denoted by $Hol_d(CP^1 \to G_r(C^n))$, the space of instantons of degree d. The intersection numbers on $Hol_d(CP^1 \to G_r(C^n))$ will be computed using the deformed potential $\tilde{W}_{n+1}(x_1, \dots, x_r) = W_{n+1}(x_1, \dots, x_r) + (-1)^r q x_1$ that reproduces the quantum cohomology $H_q^*(G_r(C^n), C) = C[x_1, \dots, x_r, q]/(\partial \tilde{W}_{n+1}/\partial x_1, \dots, \partial \tilde{W}_{n+1}/\partial x_r)$ [9]. This means that we will use formally the same formula for the intersections on the Grassmannian carried out in Section 5, however, the objects inserted in the correlators are the pullbacks of the cohomology classes (Chern classes) to the parameterizing space of holomorphic maps of degree d. These will be denoted again by $x_i (1 \le i \le r)$ such that the total power of the product of these classes is the dimension of $Hol_d(CP^1 \to G_r(C^n))$, which is, $r(n-r) + nd$ [16]. We will see that the intersection numbers on $Hol_d(CP^1 \to G_2(C^n))$ generate alternating Fibonacci numbers for $n = 5$, the Pell numbers for $n = 6$, and for $n \ge 7$ the intersection numbers generate a random walk of a particle on a line with absorbing barriers [10,11]. The self-intersection formula for $\langle x_1^{2(n-2)+nd} \rangle$, which is a special case of our two-point function given below on $Hol_d(CP^1 \to G_2(C^n))$ agrees with that computed in [6] for $n = 5$. We have also checked the geometrical meaning of the quantum correction [17] associated with the topological σ-model on CP^1 with values in the Grassmannian $G_r(C^n)$, in which computing correlators on $Hol_d(CP^1 \to G_r(C^n))$ is equivalent to doing computations on $Hol_{d-1}(CP^1 \to G_r(C^n))$ provided we set $x_r y_{n-r} = 1$.

In the following, we first write the correlators on $Hol_d(CP^1 \to G_r(C^n))$ in terms of the Chern roots q_i [9,18,19], then we will compute explicitly the formula for the intersection numbers for $r = 2, 3$. The computations on Hol_d $(CP^1 \to G_3$ $(C^n))$ are lengthy we will only give the final formula. The correlators on $Hol_d(CP^1 \to G_r\ (C^n))$ are given by

$$\langle x_1^{r(n-r)+nd-2k_1-\cdots-rk_{r-1}} x_2^{k_1} \cdots x_r^{k_{r-1}} \rangle = (-1)^{r(r-1)/2} \sum_{d\tilde{W}_{n+1}=0} \frac{x_1^{r(k-r)+nd-2k_1-\cdots-rk_{r-1}} x_2^{k_1} \cdots x_r^{k_{r-1}}}{h}, \tag{18}$$

where the summation is over a finite number of critical points of $\tilde{W}_{n+1}(x_1, \ldots, x_r)$ and $h = \det(\partial_i \partial_j \tilde{W})$. In terms of the Chern roots q_i, the potential is given by

$$\tilde{W}_{n+1}(q_i) = \sum_{i=1}^{r} \frac{q_i^{n+1}}{n+1} + (-1)^r q_i. \tag{19}$$

The Hessian in terms of the q_i's on the critical points [9] is

$$\det\left[\frac{\partial \tilde{W}_{n+1}}{\partial q_i \partial q_j}\right]_{d\tilde{W}_{n+1}=0} = \det\left[\frac{\partial^2 \tilde{W}}{\partial x_i \partial x_j}\right] \Delta^2, \tag{20}$$

where $\Delta = \prod_{j<k}(q_j - q_k)$ is the Vandermond determinant which is the Jacobian for the change of variables from q_i to x_j. Therefore, the Hessian in terms of the Chern roots is given by

$$h(q_1, \ldots, q_r) = \det\left[\frac{\partial^2 \tilde{W}}{\partial x_i \partial x_j}\right] = \frac{n^r (q_1, \ldots, q_r)^{n-1}}{\Delta^2}. \tag{21}$$

Since the Vandermond determinant vanishes for $q_i = q_j$, the summation over the critical points given by Eq. (18) involves only distinct roots $q_i (1 \le i \le r)$ of the polynomial of degree n, of the form $d\tilde{W}_{n+1} = x^n + (-1)^r$ and hence the product of the roots satisfy the identity $(q_1 \cdots q_r)^n = 1$. By using the facts $q_i^n = -1, i = 1, 2, x_1 = q_1 + q_2, x_2 = q_1 q_2$ for $r = 2$ and making the change of variables $q_i = \omega \xi_i$ with $\omega^n = -1, \xi_i^n = 1$, the two-point functions on $Hol_d(CP^1 \to G_2(C^n))$, that we denote by $I_k^{n,d}$, in terms of the new variables ξ_i are

$$\begin{aligned}\langle x_1^{2(n-2)+nd-2k} x_2^k \rangle &= -\sum_{d\tilde{W}=0} \frac{x_1^{2(n-2)+nd-2k} x_2^k}{h} \\ &= \frac{(-1)^{d+1}}{2n^2} \sum_{\xi_i^n=1, \xi_1 \ne \xi_2} [(\xi_1 + \xi_2)^2 \\ &\quad - 4\xi_1\xi_2](\xi_1 + \xi_2)^{2(n-2)+nd-2k} (\xi_1\xi_2)^{k+1},\end{aligned} \tag{22}$$

where a factor $\frac{1}{2}$ was inserted in order to avoid overcounting, since the x_i's are symmetric in the q_i's. The restriction $\xi_1 \ne \xi_2$ can be lifted provided we subtract from the sum terms with $\xi_1 = \xi_2$. In our case these terms do not contribute, therefore, we obtain

$$\langle x_1^{2(n-2)+nd-2k} x_2^k \rangle = \frac{(-1)^{d+1}}{2} \frac{1}{n^2} \sum_{\xi_i^n=1} [(\xi_1 + \xi_2)^2 - 4\xi_1\xi_2](\xi_1 + \xi_2)^{2(n-2)+nd-2k} (\xi_1\xi_2)^{k+1}. \tag{23}$$

If we set $z=\xi_1\xi_2^{-1}$ in Eq. (23), then the above summation will be over a single nth root of unity z, i.e.,

$$\begin{aligned}\langle x_1^{2(n-2)+nd-2k}x_2^k\rangle &= \frac{(-1)^{d+1}}{2}\frac{1}{n^2}\sum_{z^n=1}[(1+z)^2-4z](1+z)^{2(n-2)+nd-2k}(z)^{k+1}\\ &= \frac{(-1)^{d+1}}{2}\frac{1}{n}\sum_{z^n=1}\left[\sum_{\ell\geq 0}\binom{2n-2+nd-2k}{\ell}z^{\ell+k+1}\right.\\ &\quad\left.-4\sum_{\ell'\geq 0}\binom{2n-4+nd-2k}{\ell'}z^{\ell'+k+2}\right].\end{aligned}\tag{24}$$

The summations of $z^{\ell+k+1}$, $z^{\ell'+k+2}$ over the nth roots of unity are non-vanishing only if [2] $\ell+k+1=nq$, $\ell'+k+2=nq'$. Finally, explicit computation yields

$$\begin{aligned}&\langle x_1^{2(n-2)+nd-2k}x_2^k\rangle\\ &\quad=\frac{(-1)^{d+1}}{2}\sum_{q\in\{1,2,\ldots\}}\left[\binom{2n-2+nd-2k}{qn-(k+1)}-4\binom{2n-4+nd-2k}{qn-(k+2)}\right].\end{aligned}\tag{25}$$

If we set $k=0$ in the above formula, then we obtain the explicit formula for the self-intersection on $Hol_d(CP^1\to G_2(C^n))$, on the other hand, setting $d=0$ gives the two-point functions on $G_2(C^n)$ obtained in the previous section (Eq. (17)). Using conformal field theory, an expression for the self-intersection on $Hol_d(\Sigma_g\to G_2(C^5))$ was obtained [6], where Σ_g is a Riemann surface of genus g. When the genus $g=0$, this formula can be written as follows:

$$\begin{aligned}F(0,d) &= \frac{1}{\sqrt{5}}\left[\left(\frac{\sqrt{5}+1}{2}\right)^{5(d+1)}+(-1)^d\left(\frac{\sqrt{5}-1}{2}\right)^{5(d+1)}\right]\\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{\sqrt{5}+1}{2}\right)^{5(d+1)}-\left(\frac{1-\sqrt{5}}{2}\right)^{5(d+1)}\right],\end{aligned}\tag{26}$$

which is the well-known Binet's formula for the Fibonacci numbers $F_{5(d+1)}$ [20]. We have checked for many values of d that this formula agrees with ours, and therefore we should have the following mathematical identity on $Hol_d(CP'\to G_2(C^5))$:

$$\langle x_1^{6+5d}\rangle=\frac{(-1)^{d+1}}{2}\sum_{q\in\{1,2,\ldots\}}\left[\binom{8+5d}{5q-1}-4\binom{6+5d}{5q-2}\right]=F_{5(d+1)}.\tag{27}$$

By an explicit computation for the two-point functions on $Hol_d(CP^1\to G_2(C^5))$, see

[2] We have used the identity, $\sum_{z^n=1}z^r=n$, if $r\equiv 0\,\mathrm{mod}(n)$, and vanishing otherwise.

Table 3
Intersection numbers $I_k^{n,d}$ for $n = 4$, $d = 3, 4$; $n = 5$, $d = 2, 3$ and $n = 6$, $d = 2, 3$

k	$I_k^{4,3}$	$I_k^{4,4}$	$I_k^{5,2}$	$I_k^{5,3}$	$I_k^{6,2}$	$I_k^{6,3}$
0	128	512	610	6765	9842	265720
1	64	256	233	2584	3281	88573
2	32	128	89	987	1094	29524
3	16	64	34	377	365	9841
4	8	32	13	144	122	3280
5	4	16	5	55	41	1093
6	2	8	2	21	14	364
7	1	4	1	8	5	121
8	0	2	1	3	2	40
9		1		1	1	13
10		1		0	1	4
11						1
12						0

Table 4
Intersection numbers $I_k^{n,d}$ for $n = 7$, $d = 1, 2$ and $n = 8, 9, 10$, $d = 1$

k	$I_k^{7,1}$	$I_k^{7,2}$	$I_k^{8,1}$	$I_k^{9,1}$	$I_k^{10,1}$
0	2380	147798	15504	100947	657800
1	728	45542	4488	28101	177859
2	221	14041	1288	7752	47562
3	66	4334	364	2108	12597
4	19	1341	100	560	3264
5	5	413	26	143	820
6	1	131	6	34	196
7	0	42	1	7	43
8		14	0	1	8
9		5		0	1
10		2			0
11		1			
12		0			

Table 3 for various k and for fixed d, one can see that we should have the identity

$$\langle x_1^{6+5d-2k} x_2^k \rangle = F_{5(d+1)-2k}. \tag{28}$$

The intersection numbers given by $F_{5(d+1)-2k}$ correspond to the alternate Fibonacci numbers for $0 \leq k \leq [\frac{5}{2}(d+1)]$ with d fixed, and for $k = [\frac{5}{2}(d+1)]$ the intersection numbers are equal to 1 or 0 depending on whether the degree of the holomorphic maps d is even or odd. This seems to hold for every n. For $n = 4$, the intersection numbers $Hol_d(CP^1 \to G_2(C^4))$ are powers of 2, as one can see from Table 3. For $n = 6$, we obtain two possible sequences of Pell numbers [10], when d is odd the general term is $\frac{1}{2}(3^m - 1)$, and the other sequence given by $\frac{1}{2}(3^m + 1)$ for even degree. In general, for $n \geq 7$, the intersection numbers generate a random walk with absorbing barriers [5,10]. This is a one-dimensional random walk, in which the particle starts at point 1 and arrives eventually at the point n, the particle may never visit 0, i.e., the points 0 and n are absorbing barriers this happens when the degree is odd. When the degree d is even, the intersection numbers generate a random walk on a line for a particle that starts at point $n - 1$ (see Table 4).

4. The correlators on $Hol_d(CP^1 \to G_3(C^n))$

In computing all the correlators on $Hol_d(CP^1 \to G_3(C^n))$ that we denote by $I^{n,d}_{k_1,k_2}$ we follow the same technique as for the two-point functions computed in Section 3. Using Eqs. (18) and (21) and after some algebra, the correlators can be written as

$$\begin{aligned}
&\langle x_1^{3(n-3)+nd-2k_1-3k_2} x_2^{k_1} x_3^{k_2}\rangle \\
&\quad= -\frac{1}{n^3}\sum_{q_i^n=1,\, i=1,2,3} (q_1-q_2)^2(q_2-q_3)^2(q_1-q_3)^2(q_1+q_2+q_3)^{3(n-3)+nd-2k_1-3k_2} \\
&\qquad \times (q_1q_2+q_1q_3+q_2q_3)^{k_1}(q_1q_2q_3)^{k_2+1} \\
&\quad= -\frac{1}{6}\sum_{p,q\in\{1,2,\dots\}}\sum_{s,t=0}^{2}(-1)^{s+t}\binom{2}{s}\binom{2}{t}\sum_{\ell'=0}^{k_1}\sum_{\ell''=0}^{\ell'}\binom{k_1}{\ell'}\binom{\ell'}{\ell''}\frac{x!}{(x-y)!} \\
&\qquad \times \left[\frac{1}{(z)!(w)!}+\frac{1}{(z+2)!}-\frac{2}{(z+1)(w-1)!}\right],
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
w &= qn-(\ell'+k_2+t+1),\\
x &= 3(n-3)+nd-2k_1-3k_2,\\
y &= (p+q)n-(\ell'+\ell''+k_1+2k_2+4+s+t),\\
z &= pn-(k_1+k_2+s+3-\ell'').
\end{aligned} \tag{30}$$

If we set $d=0$, $k_1=k_2=0$ in the above formula, then we obtain the number

$$I^{n,0}_{0,0} = \frac{2(3(n-3))!}{(n-3)!(n-2)(n-1)!},$$

which is the self-intersection formula for $\langle x_1^{3(n-3)}\rangle$ on $G_3(C^n)$ given by Eq. (12). We also have checked that the above formula for $d=0$ gives the intersection numbers on $G_3(C^5)$ and

Table 5
Intersection numbers $I^{6,1}_{k_1,k_2}$

k_1	k_2: 0	1	2	3	4
0	2730	341	43	6	1
1	1365	171	22	3	0
2	683	86	11	1	0
3	342	43	5	0	
4	171	21	2		
5	85	10			
6	42	5			
7	21				

Table 6
Intersection numbers $I^{6,2}_{k_1,k_2}$

k_1	k_2					
	0	1	2	3	4	5
0	17476	21845	2731	342	43	5
1	87381	10923	1366	171	21	2
2	43691	5462	683	85	10	1
3	21846	2731	341	42	5	1
4	10923	1365	170	21	3	
5	5461	682	85	11		
6	2730	341	43	6		
7	1365	171	22			
8	683	86				
9	342	43				
10	171					

$G_3(C^6)$ and therefore setting $d = 0$ in Eq. (29), we obtain the formula for the intersection numbers $I^n_{k_1,k_2}$ on $G_3(C^n)$.

Let us check the implication of the geometrical meaning of the quantum correction [17] using our formula (Eq. (29)). As was mentioned in the beginning of this section, the quantum correction implies that correlators on $Hol_d(CP^1 \to G_3(C^n))$ are identical to those on $Hol_{d-1}(CP^1 \to G_3(C^n))$ provided we set $x_3 y_{n-3} = 1$. This can be seen by considering the following simple example: suppose we want to evaluate the correlator $\langle x_1^7 x_2 x_3 y_3\rangle_{d=1}$, where $y_3 = x_1^3 - 2x_1x_2 + x_3$. Then using the results indicated in Table 5, where $I^{6,1}_{1,1} = 171$, $I^{6,1}_{2,1} = 86$ and $I^{6,1}_{1,2} = 22$ we have $\langle x_1^7 x_2 x_3 y_3\rangle_{d=1} = 21$ which is $\langle x_1^7 x_2\rangle_{d=0} = I^6_{1,0}$ (see Table 2).

5. Conclusion

In Section 2, we obtained a closed formula for the two-point functions $\langle x_1^{r(n-r)-rk_{r-1}} x_r^{k_{r-1}}\rangle$ on $G_r(C^n)$ given by Eq. (16). When we set $r = 2$, two-point functions on $G_2(C^n)$ are obtained. Also, all the correlators $\langle x_1^{3(n-r)-2k_1-3k_2} x_2^{k_1} x_3^{k_2}\rangle$ on $G_3(C^n)$ are obtained by restricting our formula on the space of holomorphic maps of degree d to constant maps, i.e., $d = 0$. The closed formulas, obtained here are extensions of the Schubert formula equation (12) that computes the self-intersection $\langle x_1^{r(n-r)}\rangle$.

In Section 3, we obtained an explicit formula for the two-point functions on the space of holomorphic maps of degree d from CP^1 into $G_2(C^n)$. This formula generates well-known numbers like the Fibonacci numbers for $n = 5$ and the Pell numbers for $n = 6$ [10]. However, when $n \geq 7$ the formula generates a random walk of a particle on line with absorbing barriers [5,10] that starts at the point 1 and eventually reaches the point n, if d is odd. When d is even the particle starts at the point $n - 1$ (see Table 4 and [11]). At the moment, we do not understand this connection. It would be nice if this can be understood from both mathematics and physics.

Also on the moduli space of holomorphic maps into $G_r(C^n)$, we have computed intersection numbers using Eq. (29), these numbers follow well-organized patterns for given n. For $n = 5$, the intersection numbers are given by the Fibonacci numbers, this is expected since $G_3(C^5)$ and $G_2(C^5)$ are dual to each other. Setting $n = 6$, one obtains a pattern like that in Tables 5 and 6 for any degree d. For $n = 7$, we have found sequences of numbers such that the ratio of any consecutive numbers behave like that of L_n/F_n, the nth Lucas number by the nth Fibonacci number, which is known to be $\sqrt{5}$.

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